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DOMAIN DECOMPOSITION WITH NONMATCHING GRIDS: AUGMENTED LAGRANGIAN APPROACH

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ABSTRACT. We propose and study a domain decomposition method which treats the constraint of displacement continuity at the interfaces by augmented Lagrangian techniques and solves the resulting problem by a parallel version of the Peaceman-Rachford algorithm. We prove that this algorithm is equivalent to the fictitious overlapping method introduced by P.L. Lions. We also prove its linear convergence independently of the discretization step h, even if the finite element grids do not match at the interfaces. A new preconditioner using fictitious overlapping and well adapted to three-dimensional elasticity problems is also introduced and is validated on several numerical examples.

1. INTRODUCTION

In this paper, we are interested in the numerical solution of a second-order linear elliptic problem by a nonoverlapping domain decomposition technique. The model problem under consideration takes the standard form :

Find $u \in V_0$ such that

$$\sum_i \{a_i(u, v) - L_i(v)\} = 0, \quad \forall\ v \in V_0 , \tag{1}$$

where V_0 is the usual Sobolev space

$$V_0 = \left\{ v \in H^1(\Omega) \ , \ v = 0 \text{ on } \partial\Omega_D \right\} ,$$

defined over a given domain $\Omega = \bigcup_i \Omega_i$ of $\mathbb{R}^{\dim}$. When dealing with a basic Poisson equation, the local forms $a_i(u, v)$ (bilinear) and $L_i(v)$ are given on each subdomain Ω_i by

$$a_i(u, v) = \int_{\Omega_i} \nabla u \cdot \nabla v \, dx \ ,$$

$$L_i(v) = \int_{\Omega_i} f \cdot v \, dx + \int_{\partial\Omega_N \cap \partial\Omega_i} g \cdot v \, da .$$

For more complex linear elasticity problems, we would have instead

$$a_i(u, v) = \int_{\Omega_i} \sigma(x, \nabla u) : \nabla v \, dx \ ,$$

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$$\sigma(x, \nabla u) = A(x)(\nabla u + \nabla^t u)/2,$$

with $A(x)$ a symmetric positive fourth-order elasticity tensor.

In any case, even if Ω is partitioned into nonoverlapping subdomains Ω_i (Figure 1), problem (1) does not reduce to independent subproblems posed on each subdomain Ω_i because elements of the space V_0 are constrained to be continuous across the different interfaces $\partial\Omega_i \cap \partial\Omega_j$. Most nonoverlapping domain decomposition techniques handle this constraint by iterative substructuring methods, which reduce the original problem to an interface problem whose unknown is the trace of u on the interface, and which is solved iteratively by a preconditioned conjugate gradient method (see Bramble, Pasciak and Schatz [4, 5], Dryja, Smith and Widlund [9] and Le Tallec [17] for more details).

The purpose of this paper is to propose and study another numerical strategy which treats the constraint of displacement continuity at the interfaces by a Lagrange multiplier method. Based on augmented Lagrangian techniques, it first rewrites the original global minimization problem as a saddle-point problem and then solves it by a standard saddle-point algorithm which only involves the solution of local subproblems. This turns out to be equivalent to the fictitious overlapping method introduced in [20] and can be proved to converge linearly independently of the discretization step h, even if the finite element grids do not match at the interfaces.

A key point in this algorithm is the choice of the scalar product to be used on the interface.

Three different choices will be investigated, both from a mathematical and numerical point of view :

- the L^2 scalar product, which is the simplest but which leads to an h-dependent algorithm,
- a $\triangle^{-1/2}$ scalar product, easy to implement in 2D problems with straight interfaces,
- a new preconditioner using fictitious overlapping and well adapted to three-dimensional elasticity problems.

The paper is organized as follows. The continuous problem, the basic Lagrangian formulation and algorithm are introduced in §2. Convergence results are derived in §3 for the continuous problem and in §4 for its Finite Element approximation. A new preconditioner is defined in §5, and the paper concludes by several three-dimensional numerical calculations, which illustrate the performance of the proposed method and compare them with those obtained by iterative substructuring techniques.

2. Lagrangian approach of the continuous problem

2.1. **Notation.** For simplicity, the domain Ω is decomposed into two nonoverlapping subdomains Ω_i with interface S. We now introduce the boundaries (see Figure 1)

$\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$,	external Dirichlet and Neumann boundaries,
$\partial\Omega_{D_i} = \partial\Omega_D \cap \partial\Omega_i$,	local Dirichlet boundary,

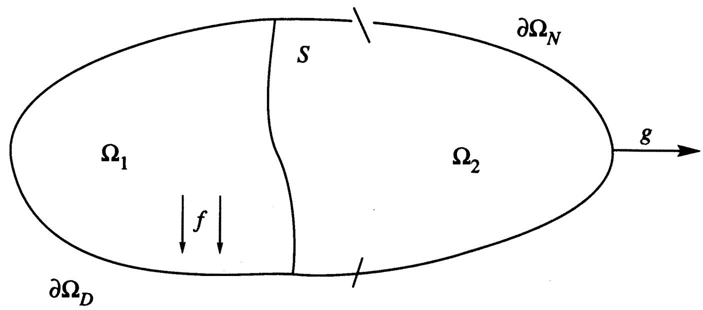

FIGURE 1. The physical problem

together with the spaces

$$V_i = \left\{ v \in H^1(\Omega_i) \;,\; v = 0 \text{ on } \partial\Omega_{D_i} \right\} \;,$$

$$V = V_1 \times V_2 \qquad \text{with norm} \quad \|\cdot\|_V = \left(\sum_i \|\cdot\|^2_{V_i} \right)^{1/2} \;,$$

$$V_0 = \{(v_1 \,,\, v_2) \in V \;,\; \operatorname{Tr} v_1|_S = \operatorname{Tr} v_2|_S\} \;,$$

$$W = \operatorname{Tr} V_1|S \;=\; \operatorname{Tr} V_2|S \,.$$

2.2. **Lagrangian formulation.** Without the linear constraint appearing in the definition of V_0, we would be faced with two independent problems posed on Ω_1 and Ω_2, respectively. To preserve this splitting property, a natural idea is then to treat the constraints $v_1|_S = v_2|_S$ by augmented Lagrangian techniques (Fortin and Glowinski [12]), that is, by penalization (one adds a term $\frac{r}{2}\|v_i - q\|^2_{H^{\frac{1}{2}}(S)}$ to the energy) and by dualization (one introduces Lagrange multipliers λ_i of the linear constraints $v_1|_S = v_2|_S = q$).

For this purpose, we introduce:

• an arbitrary scalar product on the interface space W (tentatively equivalent to the $H^{\frac{1}{2}}(S)$ scalar product) given by

$$(2) \qquad (q \,,\, \hat{q}) \;=\; \langle \mathcal{S} q \,,\, \hat{q} \rangle \,, \qquad \forall\, q \in W \,,$$

with $\mathcal{S}$ a given positive selfadjoint operator defined from W into W', and $\langle .\,,.\rangle$ the corresponding duality pairing between $H^{\frac{1}{2}}(S)$ and its dual,

• the augmented Lagrangian

$$(3) \quad \mathcal{L}_r(v \,,\, q \,,\, \lambda) \;=\; \sum_{i=1}^{2} \left\{ \frac{1}{2} a_i(u_i \,,\, v_i) - L_i(v_i) + \frac{r}{2}\|v_i - q\|^2 + (\lambda_i \,,\, v_i - q) \right\}$$

with r a given arbitrary strictly positive number.

With the notation (2)-(3), we also introduce the transformed problem:

Find $(u, q, \lambda) \in V \times W \times H$ such that

$$(4) \qquad \begin{cases} \text{(i)} \quad \frac{\partial \mathscr{L}_r}{\partial v}(u, q, \lambda) \cdot w = 0, & \forall w \in V = V_1 \times V_2, \\ \text{(ii)} \quad \langle \frac{\partial \mathscr{L}_r}{\partial q}(u, q, \lambda), dq \rangle = 0, & \forall dq \in W = \operatorname{Tr} V, \\ \text{(iii)} \quad \langle \frac{\partial \mathscr{L}_r}{\partial \lambda}(u, q, \lambda), d\lambda \rangle = 0, & \forall d\lambda \in H = W^2, \end{cases}$$

or in more details:

- equation in u

$$(5) \qquad \sum_{i=1}^{2} \{a_i(u_i, w_i) - L_i(w_i) + r\langle \mathscr{S}(u_i - q), w_i \rangle + \langle \mathscr{S}\lambda_i, w_i \rangle\} = 0, \quad \forall w \in V = V_1 \times V_2,$$

- equation in q

$$(6) \qquad \sum_{i=1}^{2} \{-r\langle \mathscr{S}(u_i - q), dq \rangle - \langle \mathscr{S}\lambda_i, dq \rangle\} = 0, \quad \forall dq \in W = \operatorname{Tr} V_i,$$

- equation in λ

$$(7) \qquad \sum_{i=1}^{2} \langle \mathscr{S}(u_i - q), d\lambda_i \rangle = 0, \quad \forall d\lambda \in H = W^2.$$

Remark 2.1. All the techniques introduced in this paper can be extended to a multidomain partition of Ω into $\Omega = \bigcup_i \Omega_i$ with interfaces $S = \bigcup_{i<j} \partial\Omega_i \cap \partial\Omega_j = \bigcup_{i<j} S_{ij}$. In this case, the global space V_0 and trace space W would be

$$V_0 = \left\{ (v_i)_i \in \prod_i V_i, \ \operatorname{Tr} v_i = \operatorname{Tr} v_j \text{ on } S_{ij}, \ \forall i < j \right\},$$

$$W = \prod_{i<j} W_{ij}, \qquad W_{ij} = \operatorname{Tr} V_i | S_{ij} = \operatorname{Tr} V_j | S_{ij}.$$

In such a treatment of interfaces, edges and corners are neglected. This is legitimate, both at the continuous and the finite element level, if there are no edges or corners (partition in strips) or if the interfaces are discretized by mortar elements (§4.2) which define discrete traces Tr_{ih} on faces and not on corners. This treatment can also be extended to general conforming partitions simply by considering any given edge separating, say, four subdomains Ω_i, Ω_j, Ω_k, Ω_l as three distinct faces S_{ij}, S_{jk} and S_{kl}.

Remark 2.2. In what follows, unless explicitly stated, the space W is endowed with the norm

$$\| w \|_W = \| \operatorname{Ext}(w) \|_{H^{1/2}(S \cup \partial\Omega_D)}$$

with $\operatorname{Ext}(w)$ the function which is equal to w on S and which is equal to zero on $\partial\Omega_D$. With this choice of norm, the trace is a continuous surjection from V_i onto W. We will refer to this norm $\| \cdot \|_W$ and to the associated scalar product as the $H^{1/2}(S)$ norm and scalar product. Strictly speaking, this terminology is correct only if the distance between $\partial\Omega_D$ and S is strictly positive.

Remark 2.3. The simplest choice for $\mathcal{S}$ consists in choosing

$$(8) \qquad (q, \hat{q}) = \langle \mathcal{S} q , \hat{q} \rangle = \int_S q \hat{q} \, dx .$$

Unfortunately, this L^2 scalar product is not equivalent to the $H^{1/2}$ scalar product and this has some negative effects on the convergence of the algorithms. Another choice is to use $\mathcal{S} = (-\Delta_S)^{1/2}$, where $-\Delta_S$ stands for the Laplace - Beltrami operator on the interface S. For this choice, the scalar product

$$(q, \hat{q}) = \langle (-\Delta_S)^{1/2} q , \hat{q} \rangle$$

is equivalent to the $H^{1/2}$ scalar product. We recall that for a straight face S perpendicular to $0x_3$, we have

$$\Delta_S (q) = \frac{\partial^2 q}{\partial x_1^2} + \frac{\partial^2 q}{\partial x_2^2} ,$$

$$(-\Delta_S)^{1/2}(q) = \sum_j \sqrt{-\lambda_j} \left(\int_S q(x) e_j(x) \, dx \right) e_j(x) .$$

Here, $(e_j)_j$ is an orthonormal basis of $L^2(S)$ composed of eigenvectors of Δ_S in W and λ_j is the eigenvalue associated with e_j. Unfortunately, this operator is nonlocal and is therefore difficult to handle numerically. A third choice will be presented later.

Remark 2.4. In the simplest case where $\mathcal{S}$ is given by (8) and where a_i is associated with a Poisson equation, the transformed problem (5)-(7) is simply

$$\begin{aligned} -\Delta u_i &= f \text{ on } \Omega_i , \\ r(u_i - q) + \lambda_i + \frac{\partial u_i}{\partial n_i} &= 0 \text{ on } S, \\ q &= \frac{1}{2}(u_1 + u_2) + \frac{1}{2r}(\lambda_1 + \lambda_2) , \\ u_i &= q . \end{aligned}$$

Observe in all cases that both subdomains play the same role.

2.3. **Equivalency result.** We have

Theorem 2.1. *The variational formulation* (1) *of the original problem is equivalent to the Lagrangian formulation* (4).

Proof. First, let $u \in V_0$ be a solution of the original problem (1). We introduce the inverse trace $\mathrm{Tr}_i^{-1} : W \to (\ker \mathrm{Tr}_i)^{\perp} \cap V_i$. By construction, Tr_i^{-1} is uniquely defined. We now define $\lambda_i \in W$ by solving the well-posed problem

$$(9) \quad (\lambda_i, \hat{q}) = \langle \mathcal{S} \lambda_i, \hat{q} \rangle = a_j(u_j, \mathrm{Tr}_j^{-1} \hat{q}) - L_j(\mathrm{Tr}_j^{-1} \hat{q}) \quad , \quad \forall \hat{q} \in W .$$

By construction, and since u is a solution of (1), we then have

$$(10) \qquad (\lambda_1 + \lambda_2, \hat{q}) = \sum_{i=1}^{2} a_i(u_i, \mathrm{Tr}_i^{-1} \hat{q}) - L_i(\mathrm{Tr}_i^{-1} \hat{q}) = 0 , \ \forall \hat{q} \in W.$$

On the other hand, setting $q = \operatorname{Tr} u$ on S, we have by definition of $\mathscr{L}_r$ and λ_i and from (1)

$$\begin{aligned}\frac{\partial \mathscr{L}_r}{\partial v_i}(u, q, \lambda) \cdot w_i &= a_i(u_i, w_i) + \langle \mathscr{S}\lambda_i, \operatorname{Tr} w_i\rangle - L_i(w_i) \\ &= a_i(u_i, w_i) + a_j(u_j, \operatorname{Tr}_j^{-1} \operatorname{Tr} w_i) - L_j(\operatorname{Tr}_j^{-1} \operatorname{Tr} w_i) - L_i(w_i) \\ &= 0 , \forall w_i \in V_i .\end{aligned}$$

By addition, this implies (4) (i). Then, by construction of $\mathscr{L}_r$, of q, and from (10), we have

$$\begin{aligned}\langle \frac{\partial \mathscr{L}_r}{\partial q}(u, q, \lambda), dq\rangle &= -\sum_{i=1}^{2}(r\langle \mathscr{S}(u_i - q), dq\rangle + \langle \mathscr{S}\lambda_i, dq\rangle) \\ &= \langle \mathscr{S}\lambda_1 + \mathscr{S}\lambda_2, dq\rangle = 0 \ \forall dq \in W,\end{aligned}$$

which is (4) (ii). Finally, by construction of q, we have

$$\langle \frac{\partial \mathscr{L}_r}{\partial \lambda}(u, q, \lambda), d\lambda\rangle = -\sum_{i=1}^{2}\langle \mathscr{S} d\lambda_i, u_i - q\rangle = 0 \quad , \quad \forall d\lambda \in H,$$

which is (4) (iii).

Conversely, let $(u, q, \lambda) \in V \times W \times H$ be a solution of (4). From (4) (iii), we first have

$$\langle \mathscr{S} d\lambda_i, u_i - q\rangle = 0 \quad , \quad \forall d\lambda_i ,$$

from which we deduce $\operatorname{Tr} u_1 = \operatorname{Tr} u_2 = q$ on the interface S. Plugging this back in (4) (ii) then yields

$$\langle \mathscr{S}(\lambda_1 + \lambda_2), dq\rangle = 0 \quad , \quad \forall dq \in W .$$

This relation used with $dq = \operatorname{Tr} w$ combined with (4) (i), and written for $w \in V_0$ finally gives

$$\begin{aligned}0 &= \frac{\partial \mathscr{L}_r}{\partial v}(u, q, \lambda) \cdot w \\ &= \sum_i a_i(u_i, w_i) + r\langle \mathscr{S}(q - u_i), w_i\rangle + \langle \mathscr{S}\lambda_i, w_i\rangle - L_i(w_i) \\ &= \sum_i a_i(u_i, w_i) - L_i(w_i), \forall w \in V_0 ,\end{aligned}$$

which is (1). □

2.4. **Solution algorithm.** Formulation (4) is particularly interesting because it can be solved by an augmented Lagrangian algorithm with good parallel properties. For example, we can use the following algorithm (called ALG3 in [12] and [16]).

Algorithm (11)-(14). With λ_i^0 and q^{-1} given, then for $n \geq 0$, with λ_i^n and q^{n-1} being given, solve successively

$$(11) \quad a_i(u_i^n, w_i) - L_i(w_i) + r\langle \mathscr{S}(u_i^n - q^{n-1}), w_i\rangle + \langle \mathscr{S}\lambda_i^n, w_i\rangle = 0, \quad \forall w_i \in V_i, \ u_i^n \in V_i, \ i = 1, 2,$$

$$(12) \quad \lambda_i^{n+\frac{1}{2}} = \lambda_i^n + r(u_i^n - q^{n-1}) \quad ; \quad i = 1, 2,$$

$$(13) \quad -r\langle \mathscr{S}(u_1^n + u_2^n - 2q^n), \hat{q}\rangle - \langle \mathscr{S}(\lambda_1^{n+\frac{1}{2}} + \lambda_2^{n+\frac{1}{2}}), \hat{q}\rangle = 0 \quad, \forall \hat{q},$$

$$(14) \quad \lambda_i^{n+1} = \lambda_i^{n+\frac{1}{2}} + r(u_i^n - q^n) \quad ; \quad i = 1, 2.$$

For the choice $\rho_n = r$, this algorithm has good convergence properties as will be proved later. Its only practical drawback concerns the choice of the operator $\mathscr{S}$ and of the coefficient r.

2.5. **Equivalence with the fictitious overlapping techniques.** By construction, we can rewrite Step 4 to 2 of algorithm (11)-(14) as follows:

$$\begin{aligned} r\mathscr{S}q^n - \mathscr{S}\lambda_i^{n+1} &= 2r\mathscr{S}q^n - \mathscr{S}\lambda_i^{n+\frac{1}{2}} - r\mathscr{S}u_i^n \\ &= r\mathscr{S}(u_j^n + u_i^n) + \mathscr{S}(\lambda_i^{n+\frac{1}{2}} + \lambda_j^{n+\frac{1}{2}}) - \mathscr{S}\lambda_i^{n+\frac{1}{2}} - r\mathscr{S}u_i^n \\ &= r\mathscr{S}u_j^n + \mathscr{S}\lambda_j^{n+\frac{1}{2}} \\ &= r\mathscr{S}u_j^n + \mathscr{S}\lambda_j^n + r\mathscr{S}(u_j^n - q^{n-1}) \\ &= 2r\mathscr{S}u_j^n + \mathscr{S}\lambda_j^n - r\mathscr{S}q^{n-1}. \end{aligned}$$

On the other hand, integrating the first step of algorithm (11)-(14) by parts, we have

$$\sigma_i^n \cdot n_i = -r\mathscr{S}u_i^n + r\mathscr{S}q^{n-1} - \mathscr{S}\lambda_i^n \quad \text{on } S.$$

After elimination of λ_i and q, there remains

$$\begin{aligned} \sigma_i^{n+1} \cdot n_i + r\mathscr{S}u_i^{n+1} &= r\mathscr{S}q^n - \mathscr{S}\lambda_i^{n+1} \\ &= r\mathscr{S}u_j^n + (r\mathscr{S}u_j^n + \mathscr{S}\lambda_j^n - r\mathscr{S}q^{n-1}) \\ &= r\mathscr{S}u_j^n - \sigma_j^n \cdot n_j. \end{aligned}$$

Therefore, Step 1 takes the final form (once integrated by parts)

$$\begin{aligned} &\operatorname{div} \sigma(\nabla u_i^{n+1}) + f_i = 0 \quad \text{on } \Omega_i, \\ &u_i^{n+1} = 0 \quad \text{on } \partial\Omega_{D_i}, \\ &\sigma_i^{n+1} \cdot n_i = g \quad \text{on } \partial\Omega_N \cap \partial\Omega_i, \\ &\sigma_i^{n+1} \cdot n_i + r\mathscr{S}u_i^{n+1} = r\,\mathscr{S}u_j^n - \sigma_j^n \cdot n_j \quad \text{on } S. \end{aligned}$$

This is precisely the nonoverlapping Schwarz alternating method proposed by P.L. Lions [20] (with $\lambda_{ij} = r\mathscr{S}$). Therefore, as observed in [16], algorithm (11)-(14) and the nonoverlapping Schwarz alternating method correspond to the same algorithm.

Remark 2.5. An alternative algorithm for solving (4) consists in eliminating u_i. The resulting dual problem in λ can then be solved by the FETI method of Farhat and Roux [11].

3. Convergence results

The solution of the original problem (1) by Algorithm (11)-(14) can now be interpreted either as the numerical integration of the associated dual problem by alternating direction methods or as the nonoverlapping Schwarz alternating method proposed by P.L. Lions [20] and studied in [15]. The first analogy appears to be very useful both from theoretical and practical points of view, since it leads to stronger convergence results in the case where $\partial\Omega_{D_i}$ is nonempty.

3.1. **Equivalence between augmented Lagrangian and alternating direction methods for the dual problem.** As seen in Glowinski and Le Tallec [16], alternating direction methods are difficult to write in a general augmented Lagrangian setting if ($B \neq \text{Id}$). Following Gabay [13], we shall overcome this difficulty by considering a dual formulation. For that purpose, let us formalize our notation and assumption.

Notation. We introduce the space E, the function $\mathscr{F}$ and the operators B, A_1 and A_2 as follows:

$$E = \{q = (q_{12}, q_{21}) \in H, q_{12} = q_{21}\},$$

$$\mathscr{F} = \text{Ind}_E = \text{Indicator function of } E \text{ in } H, \text{ with subgradient } \partial\mathscr{F} = A_1^{-1},$$

$$B: V = V_1 \times V_2 \to H = W^2,$$
$$w = (w_1, w_2) \to (\text{Tr}\, w_1, \text{Tr}\, w_2),$$

$$A_2: H \to H,$$
$$\lambda \to -Bu(\lambda),$$

with $u(\lambda)$ a solution of the domain decomposed elliptic problem

$$\sum_i (a_i(u(\lambda), w_i) - L_i(w_i)) + (\lambda, Bw) = 0 \quad , \quad \forall w \in V, u(\lambda) \in V.$$

We will assume (Assumption 3.1) that the bilinear form $\sum_i a_i(v, w)$ is coercive and continuous on the product space $V = V_1 \times V_2$. For most operators, and in particular for elasticity problems, this brings some restriction on the choice of the splitting $\Omega = \Omega_1 \cup \Omega_2$. Mainly, each domain Ω_i must be fixed on part of its boundary. Now, we are ready to use the general results of Gabay or of Glowinski and Le Tallec, which take here the form of

Theorem 3.1. *The Lagrangian formulation* (4) *is equivalent to the dual problem*

$$0 \in A_1(\lambda) + A_2(\lambda) \text{ in } H. \tag{15}$$

Proof. We have observed in Theorem 2.1 that the solution of (4) is independent of r, hence we can take $r = 0$ in (4). Then from (4) (iii), we have $\text{Tr}\, u_1 =$

$\operatorname{Tr} u_2 = q$ and hence

$$Bu = (\operatorname{Tr} u_1, \operatorname{Tr} u_2) \in E .$$

Now writing (4)(i) with $r = 0$, we get by definition of $\mathcal{L}_0$

$$\sum_i (a_i(u, w_i) - L_i(w_i)) + (\lambda, Bw) = 0 \quad , \quad \forall w \in V ,$$

or equivalently

$$-Bu = A_2(\lambda) .$$

On the other hand, writing (4) (ii) with $r = 0$ yields

$$\lambda_1 + \lambda_2 = 0 ,$$

and thus λ belongs to $E^\perp$. Since $Bu \in E$, this implies by definition of $\mathcal{F}$ and of its subgradient that

$$\lambda \in \partial \mathcal{F}(Bu),$$

or equivalently

$$Bu \in A_1(\lambda).$$

After elimination of u between the two inclusions, there results

$$-A_2(\lambda) \in A_1(\lambda) ,$$

in which we recognize (15).

Conversely, let λ be solution of (15). We first get (4) (i) by setting $u = u(\lambda)$. If we then plug the definition of u in (15), we get

$$Bu \in \partial \mathcal{F}^{-1}(\lambda),$$

which implies $\lambda \in \partial \mathcal{F}(Bu)$, that is,

$$\begin{aligned} &\lambda \in E^\perp && (\Leftrightarrow (4)\ (\text{ii})), \\ &Bu \in E && (\Leftrightarrow (4)\ (\text{iii})\ ). \end{aligned}$$

Thus, $(u, \operatorname{Tr} u, \lambda)$ is solution of (4). □

3.2. **Linear convergence of algorithm (11)-(14).**

Theorem 3.2. *Algorithm* (11)-(14) *of* §2.4 *is equivalent to the multiplicative algorithm*

(16) $$\lambda^{n+1} = (I + rA_1)^{-1}(I - rA_2)(I + rA_2)^{-1}(I - rA_1)\lambda^n .$$

Proof. We follow the steps of the general theory. By construction, algorithm (11)-(14) has the form

(17) $$\sum_i (a_i(u^n, w_i) - L(w_i)) + \langle r(Bu^n - q^{n-1}) + \lambda^n , Bw\rangle = 0, \quad \forall w \in V,$$

(18) $$\lambda^{n+\frac{1}{2}} = \lambda^n + r(Bu^n - q^{n-1}) ,$$

(19) $$\partial \mathcal{F}(q^n) \ni r(Bu^n - q^n) + \lambda^{n+\frac{1}{2}} ,$$

(20) $$\lambda^{n+1} = \lambda^{n+\frac{1}{2}} + r(Bu^n - q^n) .$$

For (19), we recall that the identity $\lambda \in \partial\mathcal{F}(q)$ is equivalent to the identities

$$q \in E \quad \text{and} \quad \lambda_1 + \lambda_2 = 0 .$$

Replacing now $r(Bu^n - q^{n-1})$ by $\lambda^{n+\frac{1}{2}} - \lambda^n$ in (17), and $r(Bu^n - q^n)$ by $\lambda^{n+1} - \lambda^{n+\frac{1}{2}}$ in (19), we get

$$u^n = u(\lambda^{n+\frac{1}{2}}) , \tag{21}$$

$$\lambda^{n+1} \in \partial\mathcal{F}(q^n) \Leftrightarrow q^n \in A_1(\lambda^{n+1}) , \tag{22}$$

respectively. In view of (21)-(22), the relations (18) and (20) now become

$$\frac{(\lambda^{n+\frac{1}{2}} - \lambda^n)}{r} + A_2(\lambda^{n+\frac{1}{2}}) + A_1(\lambda^n) \ni 0 ,$$

$$\frac{(\lambda^{n+1} - \lambda^{n+\frac{1}{2}})}{r} + A_2(\lambda^{n+\frac{1}{2}}) + A_1(\lambda^{n+1}) \ni 0 .$$

Eliminating $\lambda^{n+\frac{1}{2}}$ then leads to (16). □

This is the form introduced in [21], and on which our convergence analysis will be based.

Now, we are ready to prove the main result of this section, that is the linear convergence of algorithm (11)-(14), when written in the form (16).

Theorem 3.3. *Under Assumption* 3.1 *and if* $(\cdot , \cdot)$ *is equivalent to the* $H^{1/2}(S)$ *scalar product,* A_2 *is coercive and Lipschitz continuous on* H, *with constants* α *and* C. *Moreover, the sequence* (λ^n) *defined by* (16) *converges strongly to a solution* λ *of the dual problem* (15), *and we have*

$$|\lambda^n - \lambda|_H \leq C_0 \left(1 - \frac{4r\alpha}{(1+Cr)^2}\right)^{\frac{n}{2}} |\lambda^0 - \lambda|_H . \tag{23}$$

There also exists an optimal parameter r^* *for which we have*

$$|\lambda^n - \lambda|_H \leq C_0 \left(1 - \frac{\alpha}{C}\right)^{\frac{n}{2}} |\lambda^0 - \lambda|_H . \tag{24}$$

Proof. Step 1. By definition of A_2, we first get

$$|A_2(\lambda) - A_2(\tilde{\lambda})|_H = |Bu - B\tilde{u}|_H \leq \|B\|\|u - \tilde{u}\|_V . \tag{25}$$

On the other hand, from Assumption 3.1, we have

$$\begin{aligned}(A_2(\lambda) - A_2(\tilde{\lambda}), \lambda - \tilde{\lambda}) &= -(B(u - \tilde{u}), \lambda - \tilde{\lambda}) \\ &= -B^t(\lambda - \tilde{\lambda}) \cdot (u - \tilde{u}) \\ &= \sum_i a_i(u - \tilde{u}, u - \tilde{u}) \\ &\geq \alpha_0 \|u - \tilde{u}\|_V^2 .\end{aligned} \tag{26}$$

Using now (26) combined with (25) yields

$$\begin{aligned}\alpha_o\|u-\tilde{u}\|_V^2 &\le (A_2(\lambda)-A_2(\tilde{\lambda}),\lambda-\tilde{\lambda})\\ &\le |A_2(\lambda)-A_2(\tilde{\lambda})||\lambda-\tilde{\lambda}|\\ &\le \|B\|\|u-\tilde{u}\|_V|\lambda-\tilde{\lambda}|\end{aligned}$$

from which we deduce

$$\|u-\tilde{u}\|_V \le \frac{\|B\|}{\alpha_o}|\lambda-\tilde{\lambda}|\ ,$$

$$|A_2\lambda - A_2\tilde{\lambda}| \le \frac{\|B\|^2}{\alpha_o}|\lambda-\tilde{\lambda}|\ .$$

This is the desired Lipschitz continuity with constant $C=\|B\|^2/\alpha_0$.

To check the coercivity, we introduce

$$v-\tilde{v} \ = \ B^{-1}(\lambda-\tilde{\lambda})\ .$$

By construction of u, we then have

$$\begin{aligned}|\lambda-\tilde{\lambda}|^2 &= (BB^{-1}(\lambda-\tilde{\lambda}),\lambda-\tilde{\lambda})\\ &= (v-\tilde{v}\,,\,B^t(\lambda-\tilde{\lambda}))\\ &= -\sum_i a_i(u-\tilde{u}\,,\,v-\tilde{v})\\ &\le \|v-\tilde{v}\|_V M\|u-\tilde{u}\|_V\,,\\ &\le \|B^{-1}\|M|\lambda-\tilde{\lambda}|\|\tilde{u}-u\|_V\,.\end{aligned}$$

Here, M denotes the constant of continuity of $\sum_i a_i(.,.)$. From this, we get

$$|\lambda-\tilde{\lambda}|_H \ \le \ M\|B^{-1}\|\|u-\tilde{u}\|_V\,, \tag{27}$$

which, plugged back in (26), yields

$$\left(A_2(\lambda)-A_2(\tilde{\lambda})\,,\,\lambda-\tilde{\lambda}\right) \ \ge \ \alpha_0\|B^{-1}\|^{-2}M^{-2}|\lambda-\tilde{\lambda}|_H^2\,.$$

Hence the coercivity of A_2 with constant $\alpha = \alpha_0\|B^{-1}\|^{-2}M^{-2}$. The above proof uses the continuity of B and B^{-1}, which is a direct consequence of the trace theorem as soon as H is endowed with a $H^{\frac{1}{2}}(S)$ equivalent scalar product.

Step 2: *Convergence.* Following [21], we introduce

$$\begin{aligned}
a_2 &= A_2(\lambda)\ , & a_2^n &= A_2(\lambda^n)\ ,\\
a_1 &= -a_2\ , & &\\
\beta &= \lambda + r a_2\ , & \beta^n &= r a_2^n + \lambda^n\,,\\
\alpha &= \lambda + r a_1\ , & \alpha^n &= 2\lambda^n - \beta^n\ ,\\
a_1^n &= (\alpha^n - \beta^{n+1})/2r\,. & &
\end{aligned}$$

Since (1) has a solution, we know by equivalence that (15) has a solution λ, and hence the above quantities are well defined.

Using Proposition 1 of [21], we have

$$0 \leq (a_2^n - a_2, \lambda^n - \lambda) = \frac{1}{4r}(|\beta^n - \beta|^2 - |\alpha^n - \alpha|^2),$$

$$0 \leq \left(a_1^n - a_1, \frac{\beta^{n+1} + \alpha^n}{2} - \lambda\right) = \frac{1}{4r}(|\alpha^n - \alpha|^2 - |\beta^{n+1} - \beta|^2).$$

By addition, and from the coercivity of A_2, we deduce

$$\alpha|\lambda^n - \lambda|^2 \leq (a_2^n - a_2, \lambda^n - \lambda) \leq \frac{1}{4r}(|\beta^n - \beta|^2 - |\beta^{n+1} - \beta|^2). \tag{28}$$

On the other hand, from the Lipschitz continuity of A_2, we obtain

$$|\beta^n - \beta| = |(\lambda^n - \lambda) + r(a_2^n - a_2)| \leq (1 + rC)|\lambda^n - \lambda|.$$

Plugging this in (28) yields finally

$$\frac{1}{4r}(|\beta^n - \beta|^2 - |\beta^{n+1} - \beta|^2) \geq (1 + rC)^{-2}\alpha|\beta^n - \beta|^2,$$

that is,

$$|\beta^{n+1} - \beta|^2 \leq \left(1 - \frac{4r\alpha}{(1 + rC)^2}\right)|\beta^n - \beta|^2. \tag{29}$$

For any value of r, we thus have that β^n is a converging sequence, which converges at least linearly with constant

$$\left(1 - \frac{4r\alpha}{(1 + rC)^2}\right)^{\frac{1}{2}}.$$

This constant attains the minimum value $(1 - \alpha/C)$ for the choice $r^* = 1/C$. To deduce linear convergence of λ^n, one simply writes

$$\begin{aligned} |\lambda^n - \lambda|^2(1 + r\alpha) &\leq |\lambda^n - \lambda|^2 + r(a_2^n - a_2, \lambda^n - \lambda) \\ &\leq (\beta^n - \beta, \lambda^n - \lambda) \\ &\leq |\beta^n - \beta||\lambda^n - \lambda|, \end{aligned}$$

which yields

$$|\lambda^n - \lambda| \leq (1 + r\alpha)^{-1}|\beta^n - \beta|,$$

and hence (23) (from (29)). □

Remark 3.1. The above constants depend on the subdomain diameters and aspect ratios through the norms $\|B\|$ and $\|B^{-1}\|$. Hence, they do not scale well when the number of subdomains increases. The present version of algorithm (11)-(14) is therefore not adaptable to the many-subdomains case. □

4. Lagrangian approach of the discrete problem

4.1. **The standard conforming case.** To approximate the variational problem (1) by finite element methods, the standard procedure introduces a global finite

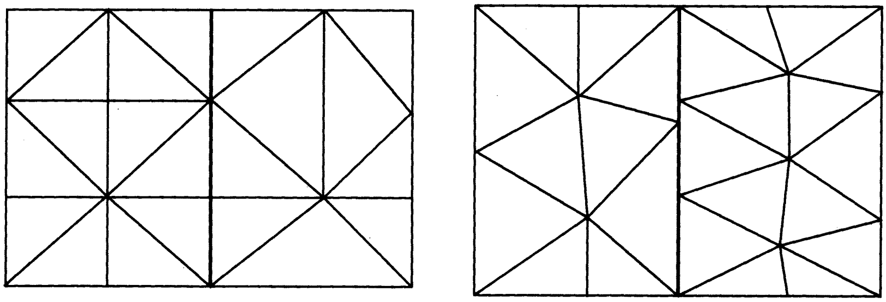

FIGURE 2. Matching and nonmatching grids

element approximation V_{0h} of V_0. If the subdomain interfaces coincide with grid lines (Figure 2: matching grids), it is then easy to define restriction spaces

$$V_{ih} = \{v_{ih} = v_{0h|\Omega_i} \ , \ v_{0h} \in V_{0h}\} \ ,$$

the product space

$$V_h = V_{1h} \times V_{2h} \ ,$$

and the trace space

$$W_h = \operatorname{Tr} V_{1h}|S = \operatorname{Tr} V_{2h}|S \, .$$

We then have as in the continuous case

$$V_{0h} = \{(v_{1h}, v_{2h}) \in V_{1h} \times V_{2h} \ , \ \operatorname{Tr} v_{1h|S} = \operatorname{Tr} v_{2h|S}\}$$

and problem (1) is approximated by

$$\sum_i \{a_i(u_h, v_h) - L_i(v_h)\} = 0 \quad , \quad \forall \ v_h \in V_{0h} \ , \ u_h \in V_{0h} \, .$$

4.2. **The discrete problem for nonmatching grids.** To approximate (1), we can also replace V_i by independent (nonmatching) Conforming Element Spaces V_{ih}, introduce a discrete auxiliary space W_h defined on the interface S (close to the so-called mortar elements [1]) and replace in the continuous problem (1) the space V_0 by its approximation V_{0h} given by

$$V_{0h} = \left\{(v_{1h}, v_{2h}) \in V_{1h} \times V_{2h} \ , \ \int_S (\operatorname{Tr} v_{1h} - \operatorname{Tr} v_{2h})\, w_h = 0 \ , \forall \ w_h \in W_h \right\} \, .$$

The discrete problem is then

$$\text{(30)} \qquad \sum_i \{a_i(u_h, v_h) - L_i(v_h)\} = 0 \ , \qquad \forall \ v_h \in V_{0h} \ , \ u_h \in V_{0h} \, .$$

We still obtain the classical equilibrium equation in V_{0h}, but this space is not classical and is not included in V_0. In other words, the continuity of the discrete solution and of the test functions at the interfaces is imposed in a weak sense only. Such an approximation may or may not be included in V_0, depending on the choice of the interface space W_h (for more details see [1] or [18]). The approximate problem (30) has already been introduced and studied in §2 of [18] or in [19], where it was proved that (30) defines a consistent nonconforming approximation of (1)

For the following, in order to get a common notation for matching and nonmatching grids, it will be convenient to introduce the discrete trace operator Tr_{ih} defined from V_{ih} into W_h and which to a given $v_{ih} \in V_{ih}$ associates its L^2 projection $\mathrm{Tr}_{ih} v_{ih}$ onto W_h. With this new notation, the space V_{0h} is then defined in both cases as

$$V_{0h} = \{(v_{1h}, v_{2h}) \in V_{1h} \times V_{2h} , \ \mathrm{Tr}_{1h} v_{1h} = \mathrm{Tr}_{2h} v_{2h}\} .$$

4.3. **Lagrangian formulation.** The augmented Lagrangian approach of §2 is able to treat matching and nonmatching grids in the same framework. For this purpose, it replaces V, W and H by the above finite-dimensional subspaces $V_h = V_{1h} \times V_{2h}$, W_h and $H_h = W_h^2$. Then, the discrete Lagrangian formulation of (30) is:

Find $(u_h, q_h, \lambda_h) \in V_h \times W_h \times H_h$ such that

$$(31) \qquad \begin{cases} \text{(i)} & \frac{\partial \mathscr{L}_r^h}{\partial v}(u_h, q_h, \lambda_h).w_h = 0 , & \forall w_h \in V_h , \\ \text{(ii)} & \langle \frac{\partial \mathscr{L}_r^h}{\partial q}(u_h, q_h, \lambda_h), dq_h \rangle = 0 , & \forall dq_h \in W_h , \\ \text{(iii)} & \langle \frac{\partial \mathscr{L}_r^h}{\partial \lambda}(u_h, q_h, \lambda_h), d\lambda_h \rangle = 0 , & \forall d\lambda_h \in H_h , \end{cases}$$

and can again be solved by Algorithm (11)-(14). Here, the augmented Lagrangian $\mathscr{L}_r^h$ is defined by

$$\mathscr{L}_r^h(v_{ih}, q_{ih}, \lambda_{ih}) = \sum_i \left\{ \frac{1}{2} a_i(v_{ih}, v_{ih}) - L_i(v_{ih}) + \frac{r}{2} |\mathrm{Tr}_{ih} v_{ih} - q_h|_h^2 \right.$$
$$\left. + \left(\lambda_{ih} , \mathrm{Tr}_{ih} v_{ih} - q_h\right)_h \right\} .$$

The space W_h is endowed with the scalar product

$$(q_h, \hat{q}_h)_h = \langle \mathscr{S}_h q_h, \hat{q}_h \rangle ,$$

with $\mathscr{S}_h$ a positive selfadjoint operator defined from W_h into W_h' and to be specified later.

Remark 4.1. The choices of r and $\mathscr{S}_h$ play no role from the theoretical point of view, but will be critical to ensure good convergence of Algorithm (11)-(14).

For the time being we make the assumptions:

Assumption 4.1. The scalar product $(\cdot, \cdot)_h$ is equivalent to the $H^{1/2}(S)$ scalar product, uniformly in h, that is, there exist constants C_1 and C_2 independent of h such that

$$C_1 \|q_h\|_W^2 \le \langle \mathscr{S}_h q_h, q_h \rangle \le C_2 \|q_h\|_W^2 , \quad \forall q_h \in W_h .$$

Assumption 4.2. The functions of W_h take zero values on $\partial\Omega_D$, and there exist strictly positive constants β_i independent of h such that

$$\inf_{q_h \in W_h} \sup_{v_{ih} \in V_{ih}} \left\{ \frac{\int_S q_h v_{ih} \, da}{\|q_h\|_{L^2(S)} \|v_{ih}\|_{L^2(S)}} \right\} \ge \beta_i .$$

The first assumption requires us again to choose a discrete scalar product which behaves like the $H^{1/2}(S)$ product. The second has already been encountered in the numerical analysis of (30) and requires that the interface space W_h not be too large with respect to the spaces $\operatorname{Tr} V_{ih}$. More precisely, this assumption means that the discrete trace Tr_{ih} is still a continuous surjection from V_{ih} onto W_h. It is automatically satisfied in the conforming case.

Assumption 4.3. The finite element space W_h appearing in Assumption 4.2 is constructed on a uniformly regular triangulation. In other words, in two-dimensional geometries, there exists a constant $C > 0$ such that, for any triangle K in $\mathcal{T}_h$ for which $K \cap \overline{S}$ is a whole edge of K, we have

$$l(K \cap \overline{S}) \geq Ch .$$

Here, $l(K \cap \overline{S})$ denotes the length of the segment $K \cap \overline{S}$.

With these assumptions, and assuming that V_{ih} is a regular finite element space in the sense of [2], we have the following preliminary lemmas, proved in the Appendix:

Lemma 4.1. *With the above assumptions, we have*

$$\| \operatorname{Tr}_{ih} v_h \|_W \leq C_3 \| v_h \|_{V_i} , \quad \forall v_h \in V_{ih} ,$$

with C_3 *independent of* h.

Lemma 4.2. *Under Assumptions* 4.1, 4.2 *and* 4.3, *the trace operator* Tr_{ih} *has an inverse* $\operatorname{Tr}_{ih}^{-1}$ *which satisfies*

$$\| \operatorname{Tr}_{ih}^{-1} w_h \|_{V_i} \leq C \| w_h \|_W , \quad \forall w_h \in W_h ,$$

with C *independent of* h.

4.4. **Uniform linear convergence of the discrete algorithm.**

Theorem 4.1. *Under Assumptions* 3.1, 4.1, 4.2 *and* 4.3, *Algorithm* (11)-(14) *applied to* (31) *converges linearly uniformly in* h, *that is,*

$$|\lambda^{n+1} - \lambda|_h \leq \left(1 - \frac{4r\overline{\alpha}}{(1+\overline{C}r)^2}\right)^{1/2} |\lambda^n - \lambda|_h .$$

Here the constants $\overline{\alpha}$ *and* $\overline{C}$ *do not depend on* h.

Proof. The proof is the same as in the continuous case when V_i is replaced by V_{ih} and Tr_i by Tr_{ih}. Hence, we first obtain the linear convergence of the algorithm with constants

$$\alpha_h = \frac{\gamma}{M^2 \|B_h^{-1}\|_h^2} ,$$
$$C_h = \frac{\|B_h\|_h^2}{\gamma} ,$$

under the notation

$$\|B_h\|_h = \sup_{v_h \in V_h} \frac{\|B_h v_h\|_h}{\|v_h\|_{1,\Omega}} ,$$

$$\|B_h^{-1}\|_h = \sup_{w_h \in W_h} \frac{\|B_h^{-1} w_h\|_{1,\Omega}}{\|w_h\|_h} .$$

But here, we have

$$B_h v_h = (\mathrm{Tr}_{1h} v_{1h} , \mathrm{Tr}_{2h} v_{2h}) .$$

Hence, we indeed have $\|B_h\|_h$ bounded independently of h as a consequence of Lemma 4.1 and Assumption 4.1 . Similarly, the boundedness of $\| B_h^{-1} w_h \|_h$ independently of h is a direct consequence of Lemma 4.2 . Therefore, the constant α_h (resp. C_h) is bounded below (resp. above) by $\overline{\alpha}$ (resp. $\overline{C}$) independently of h, and our theorem is proved. □

4.5. **A simplified choice of $\mathcal{S}_h$.** The operator $\mathcal{S}_h$ acts numerically in Algorithm (11)-(14) through the combination $\mathrm{Tr}_i^T \mathcal{S}_h \mathrm{Tr}_j$. Therefore, our first idea is to ignore the equivalence condition stated in Assumption 4.1 and to choose $\mathcal{S}_h$ in order to get the simplest possible operator $\mathrm{Tr}_i^T \mathcal{S}_h \mathrm{Tr}_j$.

To this end, using the nodal basis $(\psi_i^p)_p$ and $(\phi^l)_l$ of V_{ih} and W_h, we first define the matrices

$$B_i^{lq} = \int_S \phi^l \psi_i^q \, dx ,$$

$$M_W^{ml} = \int_S \phi^l \phi^m \, dx .$$

These "mass" matrices define the L^2 scalar product on $\mathrm{Tr}\, V_{ih} \times W_h$ and $W_h \times W_h$, respectively. With these matrices, the L^2 projection $\mathrm{Tr}_{ih} v_{ih} = \mathrm{Tr}_i V_i$ of an element v_{ih} of V_{ih} is characterized by

$$M_W \mathrm{Tr}_i V_i = B_i V_i ,$$

which means that we have

$$\mathrm{Tr}_i = M_W^{-1} B_i .$$

Now a very simple explicit choice of $\mathrm{Tr}_i^T \mathcal{S}_h \mathrm{Tr}_j$ is certainly to take

$$\mathrm{Tr}_i^T \mathcal{S}_h \mathrm{Tr}_j = B_i^T B_j$$

corresponding to the matrix

$$\mathcal{S}_h = M_W^2 .$$

This construction depends on the choice of the nodal basis, has no equivalence in terms of operators, and therefore cannot satisfy Assumption 4.1. But it is very simple, and this is the reason why we have tested it in our numerical tests.

To study the convergence of Algorithm (11)-(14) in this case, we have to estimate $\| B_h \|_h$ and $\| B_h^{-1} \|_h$. By construction, we first have for all w_h in W_h

$$\| w_h \|_h^2 = \langle M_W^2 W , W \rangle ,$$

$$\| w_h^2 \|_{L^2(S)}^2 = \langle M_W W , W \rangle ,$$

with W the vector of nodal values of w_h. Since the mass matrix M_W always has its eigenvalues in the segment $[ch^{\dim -1} , Ch^{\dim -1}]$ with c and C

independent of h, we deduce

$$c\, h^{\dim -1} \, \| w \|^2_{L^2(S)} \le \| w_h \|^2_h \le C\, h^{\dim -1} \, \| w \|^2_{L^2(S)}$$

Then, we first have

$$\begin{aligned}
\| B_h \|^2_h &= \sup_{v \in V_h} \frac{\| B_h\, v \|^2_h}{\| v \|^2_{H^1}} \\
&\le C h^{\dim -1} \sup_{v \in V_h} \frac{\| B_h\, v \|^2_{L^2(S)}}{\| v \|^2_{H^1}} \\
&\le C\, h^{\dim -1} \sup_{v \in V_h} \frac{\| \mathrm{Tr}\, v \|^2_{L^2(S)}}{\| v \|^2_{H^1}} = \mathcal{O}(h^{\dim -1}) .
\end{aligned}$$

On the other hand, from Assumption 4.2 and the inverse Sobolev inequality on W_h, we have

$$\begin{aligned}
\| B_h^{-1} \|^2_h &= \sup_{v \in \mathrm{Im}\, B_h^{-1}} \frac{\| v \|^2_{H^1}}{\| B_h\, v \|^2_h} \\
&\le \frac{C}{c\, h^{\dim -1}} \sup_{v_h} \frac{\| \mathrm{Tr}\, v \|^2_{H^{1/2}}}{\| B_h v \|^2_{L^2(S)}} \\
&\le \frac{C}{c h^{\dim}} \sup_{v_h} \frac{\| \mathrm{Tr}\, v \|^2_{L^2}}{\| B_h v \|^2_{L^2(S)}} = \mathcal{O}(\frac{1}{h^{\dim}}) .
\end{aligned}$$

Hence, the linear convergence of Algorithm (11)-(14) is obtained with constants

$$r = \frac{1}{C_h} \approx \mathcal{O}(\frac{1}{h^{\dim -1}})$$

and

$$\begin{aligned}
C' = (1 - \frac{\alpha_h}{C_h}) &= \left(1 - \frac{\|B_h^{-1}\|_h^{-2}\, \gamma^2}{\|B_h\|^2_h\, M^2} \right) \\
&\approx \left(1 - \frac{h^{\dim}\, C}{h^{\dim -1}} \right) \\
&\approx (1 - h) .
\end{aligned}$$

Remark 4.2. The choice $\mathcal{S}_h = \mathrm{Id}$ will lead to a matrix $\mathrm{Tr}_i^T \mathcal{S}_h \mathrm{Tr}_j = B_i^T M_W^{-1} B_j$ and is not practical. We could replace M_W by a diagonal lumped mass matrix but then we recover our previous choice within the factor $(\frac{1}{h})^{\dim -1}$.

5. Introduction of a fictitious Schur problem

5.1. **The fictitious Steklov-Poincaré operator.** It is worth discussing the effective choice of the operator $\mathcal{S}_h$: let us first indicate that this is by and large an open problem. In our case, some examples may be eliminated :

- the choice $\mathcal{S}_h = (-\Delta_S^h)^{1/2}$, where $-\Delta_S^h$ stands for the discrete Laplace-Beltrami operator on S, is theoretically correct but is very impractical in 3D situations;

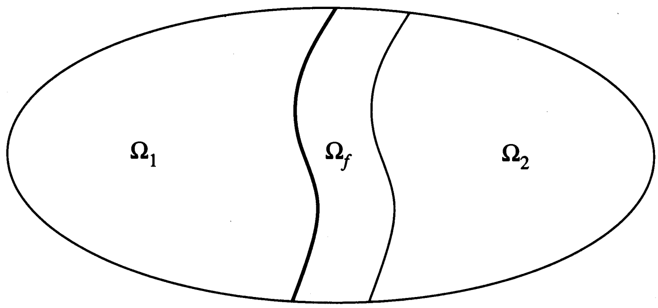

FIGURE 3

- the choice $\mathcal{S}_h = \sum_{i=1}^2 S_{ih}$, where S_{ih} is the discrete Steklov-Poincaré operator introduced in [3], is not local and is defined implicitly. This makes the problem in displacement difficult (we must invert the elasticity operator for any degree of freedom located on the interface S).

The above Steklov-Poincaré operator is not practical if the domains Ω_i are too large, but it has interesting features. Mainly, it can be defined for any geometry and for any elliptic operator, including three-dimensional anisotropic heterogeneous elasticity, and it is a coercive positive selfadjoint operator defined on the interface space W_h.

But then, for each face S, we can create in the spirit of Nepomnyaschikh [22] an artificial "dream" domain Ω_f on which to define this Steklov-Poincaré operator. Therefore, with each face S, we associate a fictitious three-dimensional domain Ω_f having S as one of its faces. This domain is to be endowed with a finite element space $H_h^1(\Omega_f)$ and with an elasticity tensor A_f.

Notation. The domain Ω is decomposed as indicated in Figure 3, the fictitious domain being denoted by Ω_f.

We now define the discrete Steklov-Poincaré operator $\mathcal{S}_h : W_h \to W_h'$ by

$$\langle \mathcal{S}_h q_h, \lambda_h \rangle := \int_{\Omega_f} \sigma_f(\nabla u_f) \cdot \nabla \operatorname{Tr}_h^{-1} \lambda_h \, dx \quad , \quad \forall \lambda_h \in W_h , \tag{32}$$

where u_f is the solution of the following Dirichlet problem:

$$\begin{cases} a_f(u_f, v_f) := \int_{\Omega_f} \sigma_f(\nabla u_f) \cdot \nabla v_f \, dx = 0 , \\ \quad \forall v_f \in H_{0h}^1(\Omega_f) = H_h^1(\Omega_f) \cap \operatorname{Ker}(\operatorname{Tr}_h) , \\ \operatorname{Tr}_h u_f = q_h \quad \text{on } S . \end{cases} \tag{33}$$

Here, $\operatorname{Tr}_h^{-1} \lambda_h$ is any function in $H_h^1(\Omega_f)$ whose weak trace is equal to λ_h. We also introduce the following space:

$$V_{ih}^{\text{ext}} = \left\{ (v_h, v_f) \in V_{ih} \times H_h^1(\Omega_f) \ , \ \operatorname{Tr}_{ih} v_h = \operatorname{Tr}_h v_f \text{ on } S \right\} . \tag{34}$$

With the above notation, we make the following assumptions.

Assumption 5.1. The bilinear form $a_f(u_f, v_f)$ satisfies the standard continuity and ellipticity conditions

$$|a_f(u_f, v_f)| \le C_1' \|u_f\|_{1,\Omega_f} \|v_f\|_{1,\Omega_f} \quad , \quad \forall u_f, v_f \in H_h^1(\Omega_f) ,$$

$$a_f(u_f, u_f) \geq C_2' \|u_f\|_{1,\Omega_f}^2 \quad , \quad \forall\, u_f \in H_h^1(\Omega_f)\,.$$

Assumption 5.2. There holds

$$\inf_{q_h \in W_h} \sup_{v_h \in H_h^1(\Omega_f)} \left\{ \frac{\int_S q_h v_h da}{\|q_h\|_{L^2(S)} \|v_h\|_{L^2(S)}} \right\} \geq \beta > 0\,.$$

We then have

Theorem 5.1. *Assumption* 4.1 *is satisfied if Assumptions* 5.1 *and* 5.2 *are.*

Proof. First, because of Assumption 5.2, Lemma 4.2 is applicable. Therefore, Tr_h is a continuous surjection from $H_h^1(\Omega_f)$ onto W_h, with a continuous inverse. Then, problem (33) can be written as

$$\begin{aligned} &a_f(u_f - \mathrm{Tr}_h^{-1} q_h\,,\, v_f) = -a_f(\mathrm{Tr}_h^{-1} q_h\,,\, v_f)\,, \\ &\forall\, v_f \in H_{0h}^1(\Omega_f)\,,\ (u_f - \mathrm{Tr}_h^{-1} q_h) \in H_{0h}^1(\Omega_f)\,. \end{aligned}$$

From Assumption 5.1, the above problem has a unique solution $u_f(q_h)$ satisfying

$$\| u_f \|_{1,\Omega_f} \leq \left(\frac{C_1'}{C_2'} + 1 \right) \| \mathrm{Tr}_h^{-1} q_h \|_{1,\Omega_f} \leq C_4' \left(\frac{C_1'}{C_2'} + 1 \right) \| q_h \|_W\,.$$

By construction, we then have

$$\begin{aligned} |\langle \mathcal{S}_h q_h\,,\, \lambda_h \rangle| &= |a_f(u_f\,,\, \mathrm{Tr}_h^{-1} \lambda_h)| \\ &\leq C_1' \| u_f \|_{1,\Omega_f} \| \mathrm{Tr}_h^{-1} \lambda_h \|_{1,\Omega_f} \\ &\leq C' \| q_h \|_W \| \lambda_h \|_W\,. \end{aligned}$$

In other words, $\mathcal{S}_h$ is a well-defined continuous operator from $H^{1/2}(S)$ in its dual. Moreover, since $\mathrm{Tr}_h\, u_f(q_h) = q_h$ by construction, we may also write

$$\langle \mathcal{S}_h q_h\,,\, \hat{q}_h \rangle = a_f(u_f(q_h)\,,\, u_f(\hat{q}_h)).$$

In this form, it is now clear that $\mathcal{S}_h$ is selfadjoint. To prove its coercivity, we rewrite this equality with $\hat{q}_h = q_h$, which implies

$$\begin{aligned} \langle \mathcal{S}_h q_h\,,\, q_h \rangle &= a_f(u_f(q_h)\,,\, u_f(q_h)) \\ &\geq C_2' \| u_f(q_h) \|_{1,\Omega_f}^2 \\ &\geq C_2' \frac{\| u_f(q_h) \|_{1,\Omega_f}^2}{\| \mathrm{Tr}_h\, u_f \|_W^2} \| \mathrm{Tr}_h\, u_f \|_W^2 \\ &\geq \frac{C_2'}{C_3'^2} \| q_h \|_W^2\,. \end{aligned}$$

This is the last estimate needed to verify Assumption 4.1 . □

We now turn to the practical solution of problem (11) in displacement when $\mathcal{S}_h$ is defined by (32). This turns out to be very simple, since we have

Theorem 5.2. *Under the above notation, the problem in displacement* (11) *takes the standard form*

$$(35)\quad \begin{cases} a_i(u_i\,,\ w_i) + r\, a_f(u_f\,,\ w_f) = L_i(w_i) \ + \langle S_h(rq^{n-1} - \lambda_i^n)\,,\, \mathrm{Tr}_{ih}\, w_i\rangle\,, \\ \qquad \forall\, (w_i\,,\, w_f) \in V_{ih}^{\mathrm{ext}}\ ,\, (u_i\,,\, u_f) \in V_{ih}^{\mathrm{ext}}\,. \end{cases}$$

This is a coupled problem posed on the union of Ω_i *and of the fictitious domain.*

Proof. First, let $u_i \in V_{ih}$. Then, with $\mathrm{Tr}_{ih}\, u_i \in W_h$ we associate u_f (extension of $\mathrm{Tr}_{ih}\, u_i$ on Ω_f), solution of (33),

$$\begin{cases} a_f(u_f\,, v_f) \ = \ 0\ ,\ \forall\, v_f \in H^1_{0h}(\Omega_f)\ , \\ \mathrm{Tr}_h\, u_f \ = \ \mathrm{Tr}_h\, u_i \qquad \text{on}\ \ S\,. \end{cases}$$

From the definition of the discrete Steklov-Poincaré operator (32), we have

$$\langle \mathcal{S}_h\, Tr_{ih}\, u_i\ ,\ \lambda_h\rangle \ = \ a_f(u_f\ ,\ \mathrm{Tr}_h^{-1}\, \lambda_h) \quad ,\quad \forall\, \lambda_h \in W_h\,.$$

Moreover, by construction $(u_i\ ,\ u_f) \in V_{ih}^{\mathrm{ext}}$.

We now rewrite (11). Replacing the expression $\langle \mathcal{S}_h \mathrm{Tr}_{ih}\, u_i\,, w_i\rangle$ by $a_f(u_f\ ,\ \mathrm{Tr}_h^{-1}\ \mathrm{Tr}_{ih}\, w_i)$, we get

$$\begin{aligned} &a_i(u_i\ ,\ w_i) + r\, a_f(u_f\ ,\ \mathrm{Tr}_h^{-1}\, \mathrm{Tr}_{ih}\, w_i) \\ &\qquad = L_i(w_i) \ + \langle S_h(rq^{n-1} - \lambda_i^n)\ ,\, \mathrm{Tr}_{ih}\, w_i\rangle\ ,\ \forall\, w_i \in V_{ih}\ , \\ &r a_f(u_f\ ,\ w_f\,) \ = \ 0\ ,\ \forall\, w_f \in H^1_{0h}(\Omega_f)\ , \\ &(u_i\ ,\ u_f) \in V_{ih}^{\mathrm{ext}}\,. \end{aligned}$$

If we add the two equalities and if we set $w \ = \ (w_i\ ,\ \mathrm{Tr}_h^{-1}\, \mathrm{Tr}_{ih}\, w_i \ + \ w_f)$, we precisely get (35).

Conversely, from (35), we get the first line of the above system by setting $w \ = \ (w_i\ , \mathrm{Tr}_h^{-1}\ \mathrm{Tr}_{ih}\, w_i)$ and we get the second line by setting $w \ = \ (0, w_f)$ with $w_f \ \in \ H^1_{0h}(\Omega_f)$. Altogether, this proves the desired equivalence result between (11) and (35). □

But now, problem (35) is easy to solve. Indeed, if we use Lagrange multipliers to enforce the weak continuity $\mathrm{Tr}_{ih}\, u_i \ = \ \mathrm{Tr}_h\, u_f$ on the interface S between Ω_i and Ω_f, (35) takes the following form:

$$(36)\qquad \begin{aligned} &a_i(u_i\ ,\ w_i) \ + \ \langle p_{ih}\ ,\ \mathrm{Tr}_{ih}\, w_i\rangle \\ &\qquad = L(w_i) + \langle S_h(rq^{n-1} - \lambda_i^n)\ ,\ \mathrm{Tr}_{ih}\, w_i\rangle \quad ,\quad \forall\, w_i \in V_{ih}\quad , \end{aligned}$$

$$(37)\qquad r a_f(u_f\ ,\ w_f) \ - \ \langle p_{ih}\ ,\ \mathrm{Tr}_h\, w_f\rangle \ = \ 0 \quad ,\quad \forall\, w_f \in H^1_h(\Omega_f) \quad ,$$

$$(38)\qquad \langle \mu_{ih}\ ,\ \mathrm{Tr}_{ih}\, u_i \ - \ \mathrm{Tr}_h\, u_f\rangle \ = \ 0 \quad ,\quad \forall\, \mu_{ih} \in W_h \quad .$$

Moreover, a straightforward manipulation yields $p_{ih} \ = \ r\mathcal{S}_h\ \mathrm{Tr}_{ih}\, u_i^n$.

Problem (36)-(38) has the same structure as the global problem proposed in §3 of [18] and can therefore be solved by the dual conjugate gradient algorithm in §3 of [18].

6. Numerical results

6.1. **Generalities.** In this section we describe some numerical results obtained with Algorithm (11)-(14). Our preconditioner is the discrete Steklov-Poincaré operator of a fictitious domain Ω_f, and the associated algorithm is compared with the unpreconditioned version ($\mathcal{S}_h = M_W^2$). This comparison is done for various mesh sizes and various numbers of subdomains in the case of matching and nonmatching grids. The numerical implementation has been done within the MODULEF Finite Element library in a multi-element and multi-problem framework. For all experiments to be described below, the stopping criterion of Algorithm (11)-(14) was

$$\frac{\| U^n - U^{n-1}\|_2}{\| U^n \|_2} < 10^{-4} .$$

In addition, the corresponding physical problem is the linear elasticity problem described in the introduction, with constitutive law

$$\sigma = \frac{E\nu}{(\nu+1)(1-2\nu)} \operatorname{Tr}\varepsilon \operatorname{Id} + \frac{E}{(\nu+1)} \varepsilon , \qquad \varepsilon = \frac{1}{2}(\nabla u + \nabla u^t) .$$

Here, E and ν are respectively the Young modulus and the Poisson coefficient.

The domain Ω is a beam of section $0.5\text{m} \times 0.2\text{m}$ and length 1m (see Figure 4). The beam is made of a quasi-incompressible material with $E = 10^{11}$MPa and $\nu = 0.49$ and is partitioned into second-order tetrahedral finite elements.

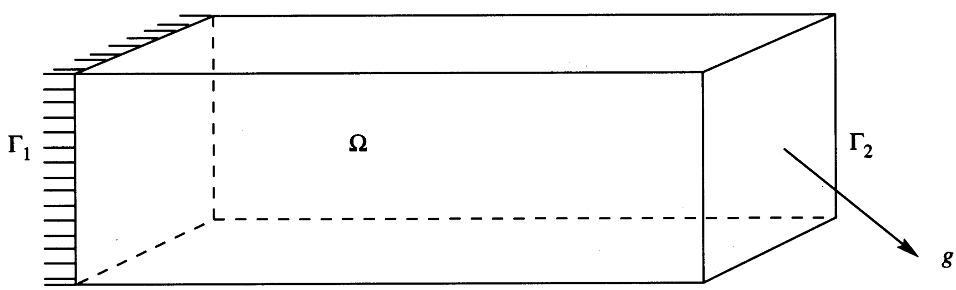

FIGURE 4. The physical configuration

Remark 6.1. The value r_{opt} (optimal value of r) in the following tables was obtained by testing by hand different possible values. The lack of an automatic strategy for the choice of r_{opt} is a limitation of the algorithm.

6.2. **Test over the discretization step** h.

Example 6.1. First, we have tested the dependency on h in the unpreconditioned version of Algorithm (11)-(14) (§4.5) in the case where the beam has been sliced along its leading dimension into two domains of equal size.

TABLE 1. Test over h with the unpreconditioned version (matching grids)
d.o.f : degrees of freedom

h^{-1}	r/E	iter.	d.o.f in Ω_1	d.o.f in Ω_2	d.o.f in S
	$0.15\ 10^4$	144			
23	$0.18\ 10^4 (r_{\text{opt}})$	98	810	810	270
	$0.2\ 10^4$	134			
	$0.1\ 10^4$	260			
46	$0.5\ 10^4 (r_{\text{opt}})$	182	5049	5049	891
	10^4	186			
	$1.5\ 10^4$	272			

TABLE 2. Test over h with the unpreconditioned version (nonmatching grids)

h^{-1}	r/E	iter.	d.o.f in Ω_1	d.o.f in Ω_2	d.o.f in S
	$0.13\ 10^4$	142			
14	$0.15\ 10^4 (r_{\text{opt}})$	120	810	525	225
	$0.2\ 10^4$	148			
	$0.5\ 10^4$	504			
28	$1.5\ 10^4 (r_{\text{opt}})$	400	5049	3159	729
	$2\ 10^4$	402			

Tables 1 and 2 show how the number of iterations and the optimal value of r depend on the parameter h, roughly showing an h^{-1} behavior. Moreover, the speed of convergence is very sensitive to the operator Tr_h^{-1}; this explains the strong increase in the number of iterations for a finer mesh in the case of nonmatching grids.

Example 6.2. Now, we consider the same examples as above, but solved with the fictitious Schur preconditioner of §5. The fictitious domain Ω_f (0.1m × 0.5m × 0.2m) is applied on the interface S and is fixed on Γ_D^f. It has the same constitutive material as the beam (Figure 5).

Tables 3 and 4 show that the preconditioned Algorithm (11)-(14) converges at a rate which is independent of r and of the mesh size. Only a slight dependence on h appears in the case of nonmatching grids.

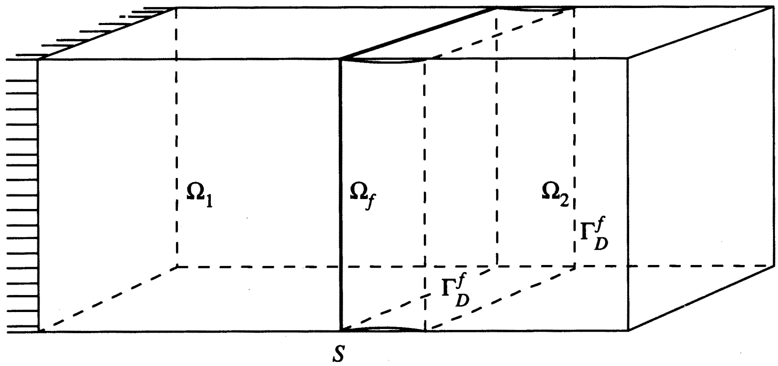

FIGURE 5. Decomposition in two subdomains

TABLE 3. Test over h in the case of matching grids (fictitious domain)

step	r	iter.	d.o.f in $\Omega_1 \cup \Omega_f$	d.o.f in $\Omega_f \cup \Omega_2$	d.o.f in S
h	0.6	64	990	990	270
	0.5(r_{opt})	60			
	0.4	74			
$h/2$	0.6	58	5643	5643	891
	0.5 (r_{opt})	56			
	0.4	66			

TABLE 4. Test over h in the case of nonmatching grids (fictitious domain)

step	r	iter.	d.o.f in $\Omega_1 \cup \Omega_f$	d.o.f in $\Omega_f \cup \Omega_2$	d.o.f in S
h	0.6	68	960	675	225
	0.5(r_{opt})	64			
	0.4	72			
$h/2$	0.6	86	5535	3645	729
	0.5 (r_{opt})	76			
	0.4	96			

Example 6.3. Here, we consider the same beam as in Example 6.2 , but decomposed into four geometrically identical subdomains (Figure 6).

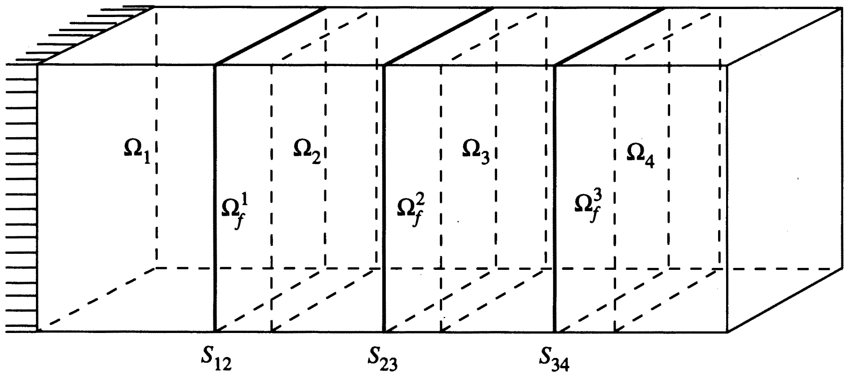

FIGURE 6. Decomposition in four subdomains

TABLE 5. Test over h in the case of matching grids (fictitious domain)

step	r	iter.	d.o.f in $\Omega_1 \cup \Omega_f^1$	d.o.f in $\Omega_2 \cup_{i=1}^{2} \Omega_f^i$	d.o.f in $\Omega_3 \cup_{i=2}^{3} \Omega_f^i$	d.o.f in $\Omega_f^3 \cup \Omega_4$	d.o.f in S
h	0.3 0.4 (r_{opt}) 0.5	92 78 90	630	810	810	630	810
$h/2$	0.3 0.4(r_{opt}) 0.5	94 88 88	3267	3861	3861	3267	2673

TABLE 6. Test over h in the case of nonmatching grids (fictitious domain)

step	r	iter.	d.o.f in $\Omega_1 \cup \Omega_f^1$	d.o.f in $\Omega_2 \cup_{i=1}^{2} \Omega_f^i$	d.o.f in $\Omega_3 \cup_{i=2}^{3} \Omega_f^i$	d.o.f in $\Omega_f^3 \cup \Omega_4$	d.o.f in S
h	0.25 0.3 (r_{opt}) 0.35 0.4	118 78 90 108	600	600	750	450	675
$h/2$	0.3 0.35 0.4 (r_{opt}) 0.45	120 110 104 114	3159	2673	3645	2187	2187

Tables 5 and 6 show that there is a slight increase in the number of iterations when we refine the mesh in the case of matching and nonmatching grids . The explanation may be the fact that we choose the same coefficient r on each subdomain (it might be better to choose different r on different subdomains).

Compared to the unpreconditioned version, CPU times and residual histories (Figures 7 and 8) show that our preconditioner turns out to be preferable when dealing with fine grids. The CPU times obtained for the Schur version also include the time required for memory swapping, which is very large for a problem of this size run on a Sun Sparc 2 workstation .

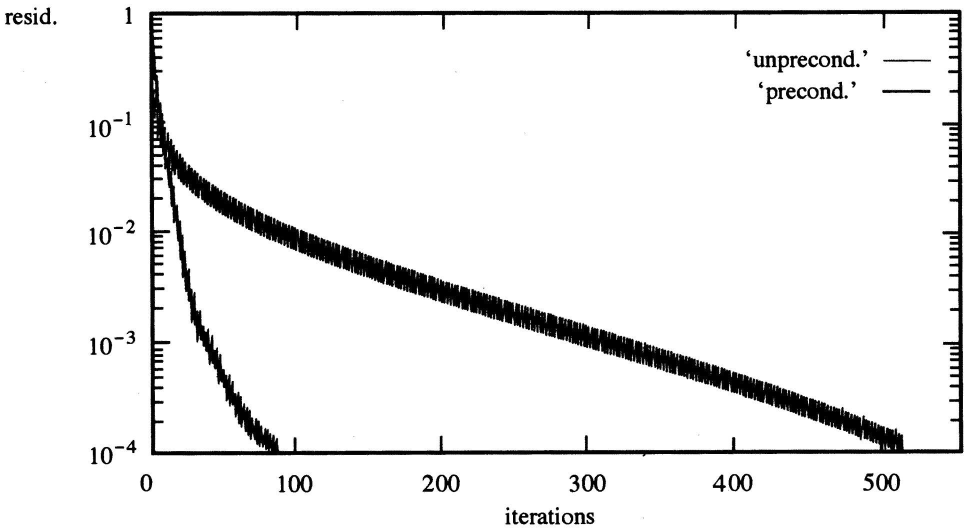

FIGURE 7. Residual against the iterations for a finer grid in the case of matching grids

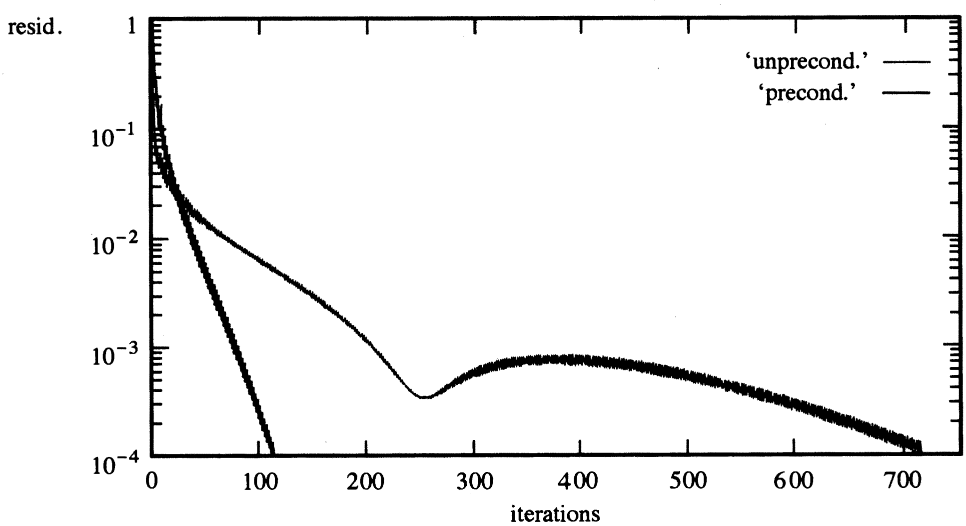

FIGURE 8. Residual against the iterations for a finer grid in the case of nonmatching grids

TABLE 7. Comparison with the unpreconditioned version for a finer grid in the case of matching grids

$\mathcal{S}_h$	iter.	residual	total approximate time
M_W^2	514	0.993 10^{-4}	2146 sec.
fictitious Schur	88	0.995 10^{-4}	1785 sec.

TABLE 8. Comparison with the unpreconditioned version for a finer grid in the case of nonmatching grids

$\mathcal{S}_h$	iter.	residual	total approximate time
M_W^2	730	0.993 10^{-4}	2310 sec.
fictitious Schur	104	0.963 10^{-4}	2250 sec.

6.3. **Comparison with Neumann-Neumann preconditioner.**

Example 6.4. Here, our domain is a beam (1m × 0.5m × 0.2m) consisting of parallel pencils. Two of them are made of a compressible material with $E_m =$ 1MPa and $\nu = 0.31$, the third is made of a quasi-incompressible material with $E_r = 10^3$MPa and $\nu = 0.49$ (see Figure 9).

In Tables 9 and 10, the CPU time on a sequential machine for Algorithm (11)-(14) with fictitious Schur preconditioner is compared to the CPU time for the Neumann-Neumann algorithm of [18] in the case of matching and nonmatching grids. The advantage of the second approach lies in the fact that it is less sensitive to the operator Tr_h^{-1} in the case of nonmatching grids. A second advantage of the second approach is that it does not require the a priori choice of a parameter r_{opt} .

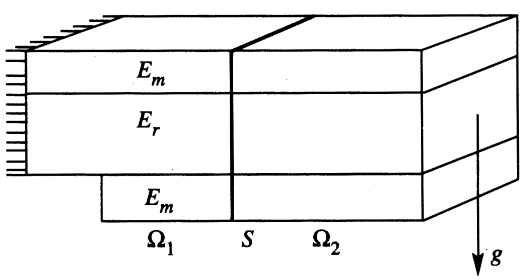

FIGURE 9

TABLE 9. Comparison with the Neumann-Neumann preconditioner in the case of matching grids

	Neumann-Neumann	fictitious Schur
d.o.f in Ω	6102	7074
assembly and factorization time	760 sec.	651 sec.
iterations	66	74
iterations time	807 sec.	765 sec.
total time	1567 sec.	1416 sec.

TABLE 10. Comparison with the Neumann-Neumann preconditioner in the case of nonmatching grids

	Neumann-Neumann	fictitious Schur
d.o.f in Ω	8100	9072
assembly and factorization time	910 sec.	691 sec.
iterations	72	131
iterations time	960 sec.	1444 sec.
total time	1870 sec.	2135 sec.

7. CONCLUSION

A Lagrangian formulation of a domain decomposed elasticity problem has been introduced and studied. For a small number of subdomains and very fine grids, this approach leads to efficient numerical algorithms, even in the case of nonmatching grids. Indeed, with the choice of adequate preconditioners such as the one introduced in §5, the method is proved to converge independently of the discretization step, which is confirmed by our numerical tests. Nevertheless, its practical implementation still faces the problem of the optimal choice of the regularization parameter r. Moreover, its convergence speed is only linear, and it does not involve any coarse grid solver. For these reasons, more classical algorithms based on preconditioned Schur complement methods might still be more competitive.

APPENDIX

Appendix 1. Proof of Lemma 4.1. By definition of Tr_{ih} (the L^2 projection operator from V_{ih} into W_h), we have for any v in $H^1(W) = \{v \in H^1(S)\ , v = 0$ on $\partial\Omega_D \cap \overline{S}\}$

$$\| \mathrm{Tr}_{ih} v \|_{0,S} \leq \| v \|_{0,S} .$$

We now denote by r_h the Clement operator introduced in P. Clement [7] and described in F. Brezzi and M. Fortin ([6, p 105]). By construction, this operator satisfies

$$\| r_h v \|_{1,S} \leq C \| v \|_{1,S} ,$$

$$\| r_h v - v \|_{0,S} \leq Ch \| v \|_{1,S} .$$

By setting $\mathrm{Tr}_{ih} v = r_h v + \mathrm{Tr}_{ih} v - r_h v$, we then have for any v in $H^1(W)$

$$\begin{aligned}
\| \mathrm{Tr}_{ih} v \|_{1,S} &\leq \| r_h v \|_{1,S} + \| \mathrm{Tr}_{ih} v - r_h v \|_{1,S} \\
&\leq \| r_h v \|_{1,S} + C h^{-1} \| \mathrm{Tr}_{ih} v - r_h v \|_{0,S} \\
&\leq \| r_h v \|_{1,S} + C h^{-1} (\| \mathrm{Tr}_{ih} v - v \|_{0,S} + \| v - r_h v \|_{0,S}) \\
&\leq \| r_h v \|_{1,S} + 2C h^{-1} \| v - r_h v \|_{0,S} \\
&\leq C \| v \|_{1,S} .
\end{aligned}$$

By interpolation between $L^2(S)$ and $H^1(W)$, we have finally

$$\| \mathrm{Tr}_{ih} v \|_W \leq C \| v \|_W . \quad \square$$

A key point in the above proof is the inverse inequality $\| w_h \|_{1,S} \leq Ch^{-1} \| w_h \|_{0,S}$ used on W_h, which requires that the triangulation on W_h must be uniformly regular.

If $\mathrm{Tr}\, V_{ih}$ is also built on a uniformly regular triangulation, then the same lemma holds also for the L^2 projection Π_{ih} onto $\mathrm{Tr}\, V_{ih}$.

Appendix 2. Proof of Lemma 4.2. From Assumption 4.2, there exists a mapping B_i^{-1} from W_h into $\mathrm{Tr}\, V_{ih}$ such that

$$\int_S B_i^{-1} w_h \mu_h \, da = \int_S w_h \mu_h \, da , \quad \forall \mu_h \in W_h ,$$

$$\| B_i^{-1} w_h \|_{0,S} \leq \frac{1}{\beta} \| w_h \|_{0,S} .$$

By extension, we will also denote by $B_i^{-1} w$ the action of B_i^{-1} on the L^2 projection of w on W_h for any $w \in L^2(S)$. We now define

$$\hat{w}_{ih} = \Pi_{ih} w_{ih} + B_i^{-1}(w_{ih} - \Pi_{ih} w_{ih}) \in \mathrm{Tr}\, V_{ih} , \quad \forall w_{ih} \in W_h ,$$

with Π_{ih} the L^2 projection operator onto $\mathrm{Tr}\, V_{ih}$. By construction, we have $\mathrm{Tr}_{ih} \hat{w}_{ih} = w_{ih}$ and

$$\| \hat{w}_{ih} \|_{1,S} \leq \| \Pi_{ih} w_{ih} \|_{1,S} + \| B_i^{-1}(w_{ih} - \Pi_{ih} w_{ih}) \|_{1,S} .$$

But, since the triangulation is uniform on V_{ih}, we have

$$\| B_i^{-1}(w_{ih} - \Pi_{ih} w_{ih}) \|_{1,S} \leq \frac{C}{h} \| B_i^{-1}(w_{ih} - \Pi_{ih} w_{ih}) \|_{0,S} .$$

Hence, it follows from Appendix 1 that $\| \hat{w}_{ih} \|_{1,S}$ satisfies

$$\begin{aligned} \| \hat{w}_{ih} \|_{1,S} &\leq \| \Pi_{ih} w_{ih} \|_{1,S} + \frac{C}{h} \| B_i^{-1}(w_{ih} - \Pi_{ih} w_{ih}) \|_{0,S} \\ &\leq C \| w_h \|_{1,S} + \frac{C}{\beta h} \| w_{ih} - \Pi_{ih} w_{ih} \|_{0,S} \\ &\leq C \| w_h \|_{1,S} + \frac{C}{\beta h} \| w_{ih} - r_h w_{ih} \|_{0,S} \\ &\leq (C + \frac{C C' h}{\beta h}) \| w_{ih} \|_{1,S} \\ &\leq C(1 + \frac{C'}{\beta}) \| w_{ih} \|_{1,S} . \end{aligned}$$

By interpolation between $H^1(W)$ and $L^2(S)$, we then obtain

$$\| \hat{w}_{ih} \|_W \leq C \| w_{ih} \|_W . \quad \square$$

We now simply define $\mathrm{Tr}_{ih}^{-1} w_h$ as the solution of the Dirichlet problem

$$\int \nabla \mathrm{Tr}_{ih}^{-1} w_h \cdot \nabla v_h \, dx = 0 , \ \forall v_h \in V_{ih}^0 , \ \mathrm{Tr}_{ih}^{-1} w_h \in V_{ih} ,$$

$$\mathrm{Tr}_{ih}^{-1} w_h = \hat{w}_{ih} \text{ on } S ,$$

posed on the space

$$V_{ih}^0 = \{ v_{ih} \in V_{ih} , \ v_{ih|S} = 0 \} .$$

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